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Hans A. Panofsky's Integral Similarity Function
—At Fifty

Gerhard Kramm[1*], Dillon J. Amaya[2], Thomas Foken[3,4], Nicole Mölders[1,5]

[1]Geophysical Institute, University of Alaska Fairbanks, Fairbanks, USA
[2]Department of Atmospheric Sciences, College of Geosciences, Texas A&M University, College Station, USA
[3]Department of Micrometeorology, University of Bayreuth, Bayreuth, Germany
[4]Member of Bayreuth Center of Ecology and Environmental Research (BayCEER), University of Bayreuth, Bayreuth, Germany
[5]Department of Atmospheric Sciences, College of Natural Science and Mathematics,
University of Alaska Fairbanks, Fairbanks, USA
Email: *kramm@gi.alaska.edu

ABSTRACT

Fifty years ago, Hans A. Panofsky published a paper entitled *Determination of stress from wind and temperature measurements*. In his famous paper, he presented a new profile function for the mean horizontal wind speed under the condition of diabatic stratification that includes his integral similarity function. With his integral similarity function, he opened the door for Monin-Obukhov scaling in a wide range of micrometeorological and microclimatological applications. In a historic survey ranging from the sixties of the past century down to the present days, we present integral similarity functions for momentum, sensible heat, and water vapor for both unstable and stable stratification, where on the one hand free convection condition and on the other hand strongly stable stratification are addressed.

1. Introduction

In his famous paper entitled *Determination of stress from wind and temperature measurements*, Hans A. Panofsky (HAP) presented a new profile function for the mean horizontal wind speed, $U(z) = |\hat{v}_H|$ with $\hat{v}_H = \hat{u}\boldsymbol{i} + \hat{u}\boldsymbol{j}$, under the condition of diabatic stratification given by [1]

$$U(z) = \frac{u_*}{\kappa}\left(\ln\frac{z}{z_0} - \Psi_m(\zeta,\zeta_0)\right). \qquad (1.1)$$

Here, $u_* = +\sqrt{|\tau|/\overline{\rho}}$ is the friction velocity, ρ is the density of air, $\boldsymbol{\tau} = -\overline{\rho u''w''}\boldsymbol{i} - \overline{\rho v''w''}\boldsymbol{j}$ is the friction stress vector invariant with height, u, v, and w are the components of the wind vector with respect to a Cartesian coordinate frame, where the horizontal unit vectors are denoted by $\boldsymbol{i}$ and $\boldsymbol{j}$, and $\boldsymbol{k}$ stands for the unit vector in vertical direction, z is the height above ground, and z_0 is the (aerodynamic) roughness length. The quantity $\Psi_m(\zeta,\zeta_0)$ is HAP's integral similarity

function for momentum defined by

$$\Psi_m(\zeta,\zeta_0) = \int_{\zeta_0}^{\zeta}\frac{1-\Phi_m(\zeta)}{\zeta}\mathrm{d}\zeta, \qquad (1.2)$$

where $\Phi_m(\zeta)$ is the local similarity function for momentum according to Monin and Obukhov [2] related to the non-dimensional shear of the mean horizontal wind speed by

$$\frac{\kappa z}{u_*}\frac{\partial U}{\partial z} = \Phi_m(\zeta). \qquad (1.3)$$

Here, $\zeta = z/L$ is the Obukhov number, L is the Obukhov stability length given by

$$L = -\frac{u_*^3}{\kappa\frac{g}{\Theta_m}\frac{H}{c_p\overline{\rho}}}, \qquad (1.4)$$

κ is the von Kármán constant, g is the acceleration of gravity, Θ_m is a potential temperature representative for

the layer under study, $H = c_p \overline{\rho w'' \Theta''}$ is the vertical component of the sensible heat flux density (hereafter a flux density is simply denoted as a flux), and c_p is the specific heat at constant pressure. The overbar denotes the conventional Reynolds' [3] mean, and the prime, as used in Equation (1.5), the departure from that. The hat characterizes Hesselberg's [4] density weighted mean and a double prime denotes the deviation thereof. According to Hesselberg, the density-weighted average of a quantity φ like the wind components, u, v, and w, the potential temperature, Θ, and the specific humidity, q, is given by $\hat{\varphi} = \overline{\rho \varphi} / \overline{\rho}$. The difference between the conventional Reynolds mean and the Hesselberg mean can be expressed by (e.g., [5-10])

$$\overline{\varphi} + \frac{\overline{\rho' \varphi'}}{\overline{\rho}} = \overline{\varphi} \left\{ 1 + \frac{\overline{\rho' \varphi'}}{\overline{\rho} \overline{\varphi}} \right\} = \hat{\varphi}, \qquad (1.5)$$

where $\overline{\varphi}$ and $\hat{\varphi}$ are nearly identical if the condition $\left| \overline{\rho' \varphi'} / \{ \overline{\rho} \overline{\varphi} \} \right| \ll 1$ is fulfilled.

Equation (1.3) can be derived on the basis of Buckingham's [11] Π theorem using the similarity hypothesis expressed by $F(z, L, u_*, \partial U / \partial z) = 0$, where it is assumed that complete similarity is established (e.g., [12, 13]). For $\Phi_m(0) = 1$ which is valid for neutral stratification, Formula (1.2) immediately provides $\Psi_m(0) = 0$, and hence, Equation (1.1) reduces to the well-known logarithmic wind profile given by

$$U(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_0}. \qquad (1.6)$$

One of the notable advantages of HAP's profile formula is obvious. Equation sets (1.1) and (1.2) represent a general solution within the framework of the physics of the Atmospheric Surface Layer (ASL), the first layer of the atmosphere of the thickness of a few decameters because the this general solution is independent of the shape of the local similarity function.

HAP used his diabatic wind profile to estimate surface stress from measured wind and temperature profiles. He showed that excellent estimates of stress can be made, given the roughness length, an estimate of the Richardson number and an accurate wind at one level. He suggested that his theory can further be applied to estimate the roughness length from relatively few observations of wind and temperature not necessarily under neutral conditions. The results of various authors support his suggestion [14-18]. With his integral similarity function HAP opened the door for Monin-Obukhov scaling in a wide range of micrometeorological and microclimatological applications. This includes the parameterization of the eddy fluxes of sensible and latent heat in energy flux budgets at the Earth's surface, and the estimation of eddy fluxes of long-lived trace gases if fast response sensors are not available.

In Section 2, we will sketch the derivation of HAP's integral similarity functions for vertical profiles of horizontal mean wind speed, potential temperature, and specific humidity. In Sections 3 and 4, a brief, but thorough presentation of integral similarity functions for unstable stratification (Section 3) and stable stratification (Section 4) published during the past four decades will be presented, starting with the results of Paulson [19] for unstable stratification. Section 5 contains our final remarks and conclusions.

2. Theoretical Background

In the following, we assume that the conditions of stationary state and horizontal homogeneity are fulfilled as required by Monin-Obukhov similarity hypothesis.

After introducing their local similarity function $\Phi_m(\zeta)$ Monin and Obukhov [2] assumed that it can be expressed by a power series. They argued that for $|\zeta| < 1$ this power series can be restricted to the first terms, *i.e.*,

$$\Phi_m(\zeta) = 1 + \beta \zeta, \qquad (2.1)$$

where $\beta \cong 0.6$. Thus, Equation (1.3) becomes

$$\frac{\kappa z}{u_*} \frac{\partial U}{\partial z} = 1 + \beta \zeta. \qquad (2.2)$$

Integrating this equation over the layer $[z_1, z_2]$ of the ASL and assuming that the friction velocity is invariant with height yield

$$U(z_2) - U(z_1) = \frac{u_*}{\kappa} \left(\ln \frac{z_2}{z_1} + \beta (\zeta_2 - \zeta_1) \right). \qquad (2.3)$$

This expression is known as the logarithmic-linear wind profile. It is usually considered for stable and slightly unstable stratification and includes the special case of neutral stratification. For deriving the temperature profile, Monin and Obukhov assumed that within the limits of meteorological observations $\Phi_h(\zeta) = \Phi_m(\zeta) = 1 + \beta \zeta$, where $\Phi_h(\zeta)$ is the local stability function for sensible heat related to the nondimensional gradient of the potential temperature by

$$\frac{\kappa z}{\Theta_*} \frac{\partial \hat{\Theta}}{\partial z} = \Phi_h(\zeta). \qquad (2.4)$$

This equation is based on the similarity hypothesis $F(z, L, \Theta_*, \partial \hat{\Theta} / \partial z) = 0$ (e.g., [13]). Here, Θ_* is the temperature scale that serves to compute H according to

$$H = c_p \overline{\rho} u_* \Theta_* = \text{const.} \qquad (2.5)$$

Since H is also considered as invariant with height, water substances must not undergo phase transition processes. Under such a condition, the integration of Equa-

tion (2.4) over the layer $[z_1, z_2]$ of the ASL yields

$$\hat{\Theta}(z_2) - \hat{\Theta}(z_1) = \frac{\Theta_*}{\kappa}\left(\ln\frac{z_2}{z_1} + \beta(\zeta_2 - \zeta_1)\right). \quad (2.6)$$

These profile function for the mean horizontal wind speed and the mean potential temperature are only valid for $|\zeta| < 1$.

A couple of years later, various authors proposed the so-called KEYPS formula[1] for the local similarity function for momentum given by

$$\Phi_m^4(\zeta) - \gamma_1 \zeta \Phi_m^3(\zeta) = 1 \quad \text{for } \zeta \leq 0. \quad (2.7)$$

The KEYPS formula with $\gamma_1 \cong 9$ has experimentally been deduced by Businger *et al.* [27] for the stability range $-2 \leq \zeta < 0$; Panofsky and Dutton [28], however, recommended: $\gamma_1 \cong 15$. Formula (2.7) indicates a $\Phi_m(\zeta) \cong (-\gamma_1\zeta)^{-1/3}$ behavior for $\zeta \ll 0$ as expected for free convective conditions.

Apparently, such a local similarity function demands a more general solution of Equation (1.3). Panofsky [1] found it by rearranging Equation (1.3) as follows:

$$\frac{\kappa z}{u_*}\frac{\partial U}{\partial z} = \Phi_m(\zeta) = 1 - 1 + \Phi_m(\zeta). \quad (2.8)$$

Integrating this equation over the layer $[z_1, z_2]$ of the ASL yields:

$$U(z_2) - U(z_1) = \frac{u_*}{\kappa}\left(\ln\frac{z_2}{z_1} - \Psi_m(\zeta_2, \zeta_1)\right), \quad (2.9)$$

where HAP's integral similarity function for momentum is given by

$$\Psi_m(\zeta_2, \zeta_1) = \int_{\zeta_1}^{\zeta_2}\frac{1 - \Phi_m(\zeta)}{\zeta}\mathrm{d}\zeta. \quad (2.10)$$

Choosing $z_1 = z_0$ provides Equations (1.1) and (1.2). As mentioned before, this equation set is the general solution that is independent of the shape of the local similarity function and the range of thermal stratification.

For the vertical profiles of the mean potential temperature and the mean specific humidity we obtain in a similar manner

$$\hat{\Theta}(z_2) - \hat{\Theta}(z_1) = \frac{\Theta_*}{\kappa}\left(\ln\frac{z_2}{z_1} - \Psi_h(\zeta_2, \zeta_1)\right) \quad (2.11)$$

and

$$\hat{q}(z_2) - \hat{q}(z_1) = \frac{q_*}{\kappa}\left(\ln\frac{z_2}{z_1} - \Psi_q(\zeta_2, \zeta_1)\right), \quad (2.12)$$

respectively. Here,

$$\Psi_{h,q}(\zeta_R, \zeta_r) = \int_{\zeta_r}^{\zeta_R}\frac{1 - \Phi_{h,q}(\zeta)}{\zeta}\mathrm{d}\zeta \quad (2.13)$$

are the integral similarity functions for sensible heat (subscript h) and water vapor (subscript q), respectively. Furthermore, q_* is the humidity scale related to the vertical component of the water vapor flux, Q, by

$$Q = \bar{\rho}u_* q_* = \text{const.}, \quad (2.14)$$

and $\Phi_q(\zeta)$ is the local similarity function for water vapor related to the non-dimensional gradient of the specific humidity by

$$\frac{\kappa z}{q_*}\frac{\partial \hat{q}}{\partial z} = \Phi_q(\zeta). \quad (2.15)$$

This equation is based on the similarity function $F(z, L, q_*, \partial\hat{q}/\partial z) = 0$ (e.g., [13]). When the transfer of water vapor across the ASL plays a notable role like over water surfaces and wet soil or vegetation, it is indispensable to use the following expression for the Obukhov stability length [29]:

$$L = -\frac{c_{p,d}\bar{\rho}u_*^3}{\kappa\frac{g}{\hat{\Theta}}\left(H + 0.61c_{p,d}\hat{\Theta}Q\right)} = \frac{u_*^2}{\kappa\frac{g}{\hat{\Theta}}\left(\Theta_* + 0.61\hat{\Theta}q_*\right)}, \quad (2.16)$$

where $c_{p,d}$ is the specific heat at constant pressure for dry air. Note that long-lived trace gases can be handled in a similar manner, where it is often assumed that the local similarity function for such trace gases, $\Phi_c(\zeta)$, is equal to that of water vapor.

As discussed by various authors [13,18,30], the local similarity functions, $\Phi_m(\zeta)$, $\Phi_h(\zeta)$, and $\Phi_q(\zeta)$, impose as universal laws for describing the surface (constant flux) layer turbulence. Reviews of field campaigns and empirical findings can be found in [26,31-36].

3. Integral Similarity Functions for Unstable Stratification

Even though the O'KEYPS formula was established for covering the entire unstable range from close to neutral stratification to free convective conditions, another local similarity function for unstable stratification was eventually proposed by Businger [37], Dyer (unpublished; see [38]), and Pandolfo [39]. It reads

$$\Phi_m(\zeta) = (1 - \gamma_2\zeta)^{-1/4}, \quad (3.1)$$

where $\gamma_2 \cong 16$ is another empirical constant. This equation is called the Businger-Dyer-Pandolfo relationship. It has experimentally been proved by Dyer and Hicks [41], Businger *et al.* [27] and others, where their results mainly covered the stability range $-2 \leq \zeta < 0$ (e.g., [28,33,35, 36,40], and **Figure 1**). From a physical point of view the

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[1]KEYPS stands for the initials of various authors who proposed this formula (Kazansky and Monin [20], Ellison [21], Yamamoto [22], Panofsky [23], and Sellers [24]). As Obukhov already suggested it in 1946, the KEYPS formula was eventually renamed in O'KEYPS formula (e.g., [13,25,26]).
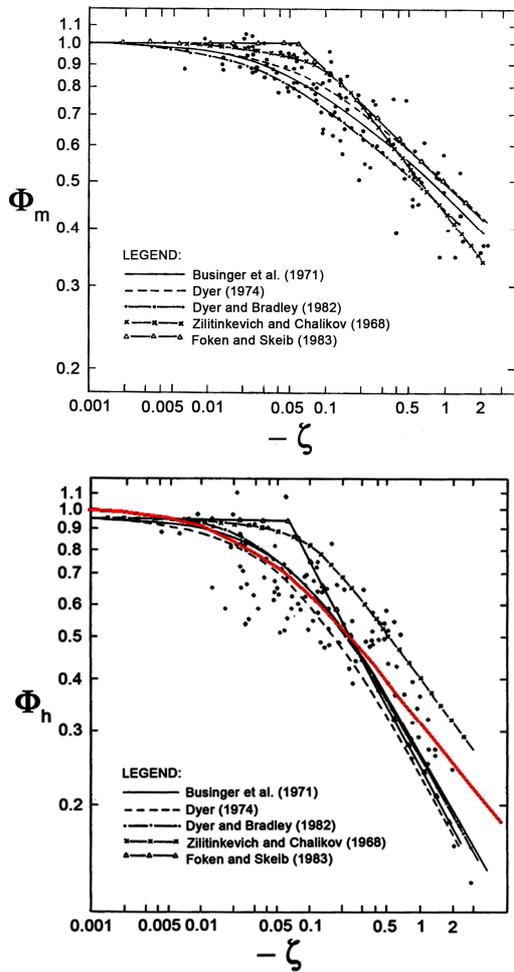
Figure 1. The local similarity functions $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ versus the Obukhov number ζ for unstable stratification (adopted from Högström [40]). The dots represent Högström's field data, and the red line in the lower part of this figure illustrates Equation (3.26). Note that the formula of Dyer (1974, [31]) is identical with that of Dyer and Hicks [41] listed here as Equations (3.1) and (3.3). The formulae of Businger *et al.* (1971, [27]) are given by Equations (3.6) and (3.7). Note that the references Dyer and Bradley (1982), Zilitinkevich and Chalikov (1968), and Foken and Skeib (1983) are listed here as references [42-44].

O'KEYPS formula seems to be more preferable than the Businger-Dyer-Pandolfo relationship because the O'KEYPS formula can be related to the local balance equation of the turbulent kinetic energy if the conditions of stationary state and horizontal homogeneity are fulfilled (e.g., [13]).

Since the O'KEYPS formula seems to be bulky, Carl *et al.* [45] and Gavrilov and Petrov [46] eventually proposed the expression

$$\Phi_m(\zeta) = (1 - \gamma_3 \zeta)^{-1/3} \tag{3.2}$$

for the stability range $-10 \leq \zeta \leq 0$, where $\gamma_3 \cong 15$ was

determined using observations from various locations. This equation reflects the same asymptotic behavior like the O'KEYPS formula, but notably differs from that of the Businger-Dyer-Pandolfo relationship (see **Figure 2**).

Businger [37] and Pandolfo [39] also suggested the following relationship between the local similarity functions for momentum und sensible heat:

$$\Phi_h(\zeta) = \Phi_m^2(\zeta) = (1 - \gamma_2 \zeta)^{-1/2}. \tag{3.3}$$

Dyer and Hicks [41] eventually proved Equation (3.3) for the stability range $-1 \leq \zeta < 0$. Since the so-called gradient-Richardson number *Ri* may be expressed by the non-dimensional gradients, *i.e.*,

$$Ri = \frac{\Phi_h(\zeta)}{(\Phi_m(\zeta))^2} \zeta, \tag{3.4}$$

the Businger-Pandolfo relationship $\Phi_h(\zeta) = \Phi_m^2(\zeta)$ for unstable stratification provides (see **Figures 3** and **4**).
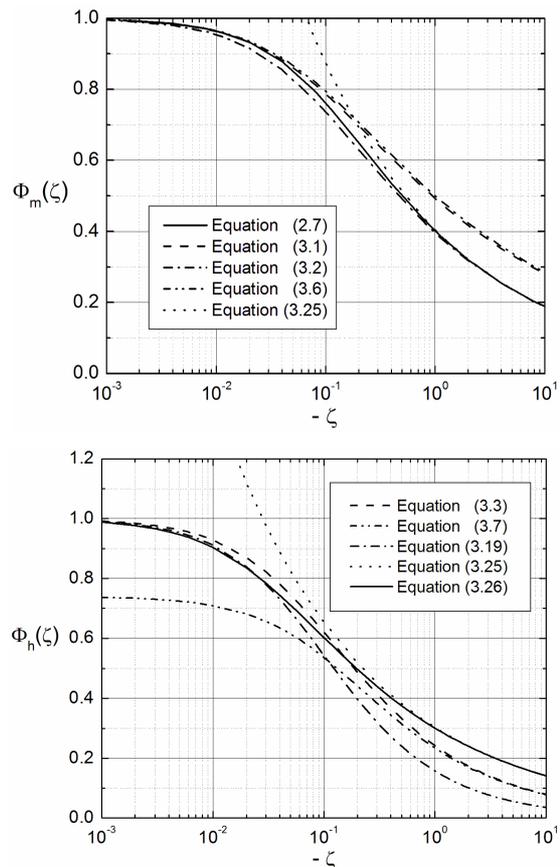
Figure 2. The local similarity functions $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ versus the Obukhov number ζ for unstable stratification. In contrast to Figure 1, this diagram shows the original ones of Dyer and Hicks [41], Equations (3.1) and (3.3), and Businger *et al.* [27], Equations (3.6) and (3.7). Note that Equation (3.25) represents free convective conditions as deduced by Prandtl-Obukhov-Priestley scaling.
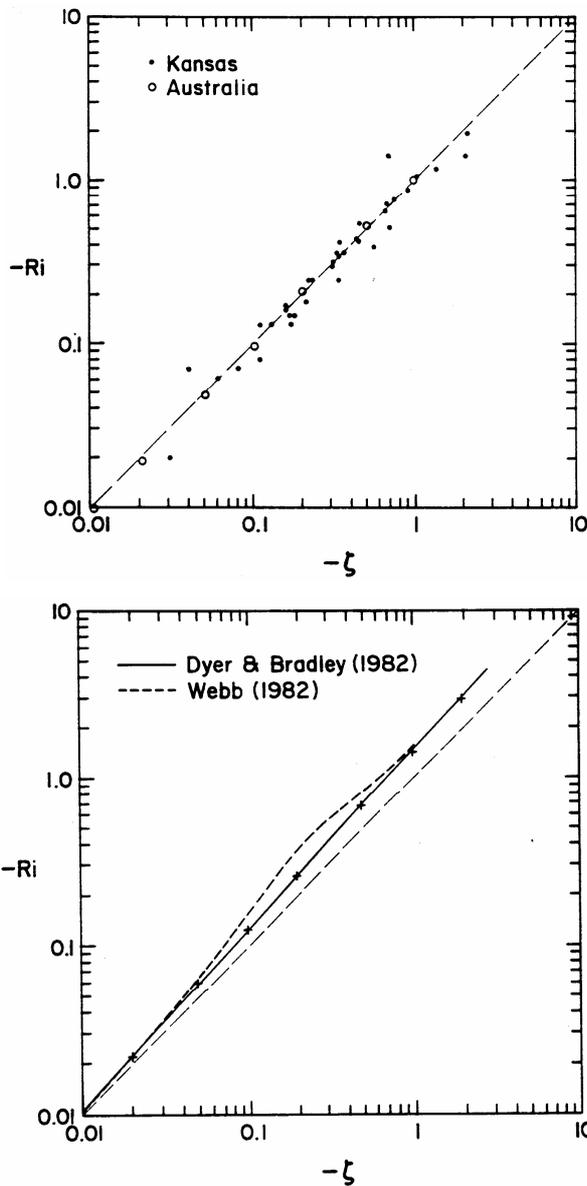
Figure 3. Gradient Richardson number, *Ri*, versus the Obukhov number ζ for three different field experiments (adopted from Businger [38]). Note that the references Dyer and Bradley (1982) and Webb (1982) are listed here as references [42,47].

$$Ri = \zeta \quad \text{for } \zeta < 0 . \tag{3.5}$$

However, based on the data of the 1968 Kansas field experiment [48], Businger *et al.* [27] suggested

$$\Phi_m(\zeta) = (1 - \gamma_4 \zeta)^{-1/4} \tag{3.6}$$

and

$$\Phi_h(\zeta) = 0.74(1 - \gamma_5 \zeta)^{-1/2} \tag{3.7}$$

for $\kappa = 0.35$, where $\gamma_4 \cong 15$ and $\gamma_5 \cong 9$ (see **Figure 2**). These local similarity functions provide
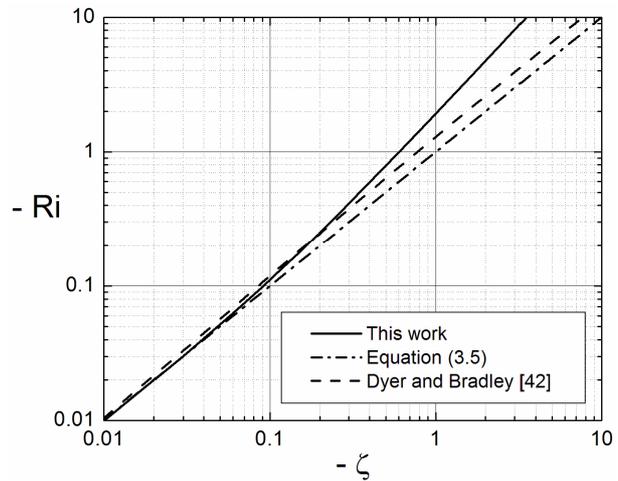
Figure 4. Gradient Richardson number, *Ri*, versus the Obkhov number, ζ, for unstable stratification, where *Ri* has been deduced on the basis of formulae (3.2), (3.4), and (3.26). The one-to-one line represents
$$\Phi_h(\zeta) = \Phi_m^2(\zeta) \Leftrightarrow Ri = \zeta \quad \text{(see Equations (3.3) to (3.5)).}$$

$$Ri = 0.74\left(\frac{1 - \gamma_4 \zeta}{1 - \gamma_5 \zeta}\right)^{\frac{1}{2}} \zeta . \tag{3.8}$$

Consequently, $|Ri| < |\zeta|$, *i.e.*, the identity $Ri = \zeta$ would be no longer valid for unstable stratification. Dyer and Bradley [42] and Webb [47] pointed out that small deviations from this identity might occur. However, as illustrated **Figure 3**, they found $|Ri| > |\zeta|$. Note that Högström [40] re-calibrated the local similarity functions of various authors listed in **Figure 1** with respect to $\kappa = 0.40$. Thus, the local similarity functions illustrated in this figure are Högström's modified versions of $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ (The same is true in case of stable stratification). A value of $\kappa = 0.40$ is, indeed, widely recommended. However, based on 553 independent determinations of κ (the largest, most comprehensive atmospheric data set ever used to evaluate the von Kármán constant) Andreas *et al.* [49] derived a value of $\kappa = 0.387 \pm 0.003$, constant for $2 \leq \eta \leq 100$, where $\eta = u_* z/\nu$ is the roughness Reynolds number, and ν is the kinematic viscosity. Frenzen and Vogel [50] also found a value of $\kappa = 0.387 \pm 0.010$, but their result is based on 29 data pairs only. Nevertheless, we do not recommend a recalibration of the local similarity functions with respect to a certain value of the von Karmán constant. Instead, $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ should be used with the original values of κ. Using, for instance, the original ones of Businger *et al.* [27] and a von Kármán constant of $\kappa = 0.40$ is a notable source of inconsistency.

At the end of the sixties, the time was ripe for determining Panofsky's integral similarity function on the

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basis of empirically determined local similarity functions. Paulson [19] was the first who derived integral similarity functions for unstable stratification represented here in the more general form for the layer $[z_1, z_2]$ of the ASL:

• O'KEYPS formula:

$$\Psi_m(\zeta_2, \zeta_1) = \Phi_m(\zeta_1) - \Phi_m(\zeta_2) + 2\ln\frac{1 + \Phi_m(\zeta_2)}{1 + \Phi_m(\zeta_1)}$$
$$+ \ln\frac{1 + \Phi_m^2(\zeta_2)}{1 + \Phi_m^2(\zeta_1)} - 3\ln\frac{\Phi_m(\zeta_2)}{\Phi_m(\zeta_1)} \qquad (3.9)$$
$$+ 2\arctan\frac{\Phi_m(\zeta_2) - \Phi_m(\zeta_1)}{1 + \Phi_m(\zeta_2)\Phi_m(\zeta_1)}$$

• Businger-Dyer-Pandolfo relationship:

$$\Psi_m(\zeta_2, \zeta_1) = 2\ln\frac{1 + y_2}{1 + y_1} + \ln\frac{1 + y_2^2}{1 + y_1^2}$$
$$- 2\arctan\frac{y_2 - y_1}{1 + y_2 y_1} \qquad (3.10)$$

and

$$\Psi_h(\zeta_2, \zeta_1) = 2\ln\frac{1 + y_2^2}{1 + y_1^2}, \qquad (3.11)$$

where

$$y_{1,2} = \Phi_m^{-1}(\zeta_{1,2}) = \left(1 - \gamma_2\zeta_{1,2}\right)^{1/4} \qquad (3.12)$$

are the reciprocal expressions of the local similarity functions for momentum at the two heights z_1 and z_2. Obviously, the O'KEYPS solution (3.9) seems to be more bulky than that obtained with the Businger-Dyer-Pandolfo relationship given by Equation (3.10). This might be the reason why the latter has been more widely used, even though the former has a stronger physical background [13].

Since Paulson [19] assumed that $\zeta_1 = 0$ and, hence, $\Phi_m(\zeta_1) = 1$, his solutions substantially agree with Equations (3.9) to (3.11), *i.e.*,

$$\Psi_m(\zeta) = 1 - \Phi_m(\zeta) + 2\ln\frac{1 + \Phi_m(\zeta)}{2}$$
$$+ \ln\frac{1 + \Phi_m^2(\zeta)}{2} - 3\ln\Phi_m(\zeta) \qquad (3.13)$$
$$+ 2\arctan\left(\Phi_m(\zeta)\right) - \frac{\pi}{2},$$

$$\Psi_m(\zeta) = 2\ln\frac{1 + y}{2} + \ln\frac{1 + y^2}{2}$$
$$- 2\arctan(y) + \frac{\pi}{2}, \qquad (3.14)$$

and

$$\Psi_h(\zeta) = 2\ln\frac{1 + y^2}{2}, \qquad (3.15)$$

where $\zeta_2 = \zeta = z/L$ and the identity

$$\arctan\frac{y - 1}{1 + y} = \arctan(y) - \arctan(1)$$

with $\arctan(1) = \pi/4$ have been used. Often, HAP's integral similarity function is written as

$$\Psi_{m,h,q}(\zeta, \zeta_0) = \Psi_{m,h,q}(\zeta) - \Psi_{m,h,q}(\zeta_0),$$

where Paulson's Equations (3.13) to (3.15) are considered for both ζ and $\zeta_0 = z_0/L$. This kind of splitting is highly awkward because it is neither reasonable nor advantageous. In addition, Paulson derived his equations for $\zeta_0 = z_0/L = 0$. Thus, the quantity $\Psi_{m,h,q}(\zeta_0)$ is always equal to zero if Paulson's Equation (3.13) to (3.15) are considered [51].

Lettau [52] eventually presented the solution for the local similarity function of Carl *et al.* [45] (see Equation (3.2)). It reads in the more general form for the layer $[z_1, z_2]$ of the ASL:

$$\Psi_m(\zeta_2, \zeta_1)$$
$$= \frac{3}{2}\ln\frac{y_2^2 + y_2 + 1}{y_1^2 + y_1 + 1} - \sqrt{3}\arctan\frac{x_2 - x_1}{1 + x_2 x_1}, \qquad (3.16)$$

where

$$y_{1,2} = \Phi_m^{-1}(\zeta_{1,2}) = \left(1 - \gamma_3\zeta_{1,2}\right)^{1/3} \qquad (3.17)$$

are the reciprocal expressions of the local similarity functions for momentum at the two heights z_1 and z_2, and

$$x_{1,2} = \left(2y_{1,2} + 1\right)\big/\sqrt{3}. \qquad (3.18)$$

Lettau's [52] solution substantially agrees with Equations (3.16) to (3.18) for $\zeta_1 = 0$ and, hence, $\Phi_m(\zeta_1) = 1$ is considered. The integral similarity functions for momentum, (3.9), (3.10), and (3.16) are illustrated in **Figure 5**. As expected, formulae (3.9) and (3.16) only differ hardly when ζ tends to Obukhov numbers much smaller than zero, *i.e.*, $\zeta \ll 0$, representing free-convective conditions. Simultaneously, the difference between Equation (3.10) and the other two formulae grows continuously. Thus, we recommend to use the integral similarity function (3.16) for practical purposes. Its results are close to those provided by the O'KEYPS solution, but the former is more convenient than the latter.

Under the assumption that the Businger-Pandolfo relationship $\Phi_h(\zeta) = \Phi_m^2(\zeta) \Leftrightarrow Ri = \zeta$ holds for the entire range of unstable stratification and that the local similarity function for momentum is given by Equation (3.2) we obtain

$$\Phi_h(\zeta) = \left(1 - \gamma_3\zeta\right)^{-2/3}. \qquad (3.19)$$

This local similarity function illustrated in **Figure 2**
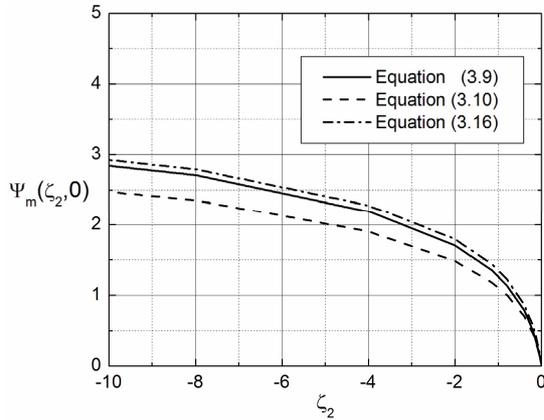
Figure 5. The integral similarity function for momentum, $\Psi_m(\zeta_2, 0)$, provided by formulae (3.9), (3.10), and (3.16) and plotted against the Obukhov number $\zeta_2 = z_2/L$.

leads to [13]

$$\Psi_h(\zeta_2, \zeta_1) = \frac{3}{2} \ln \frac{y_2^2 + y_2 + 1}{y_1^2 + y_1 + 1} + \sqrt{3} \arctan \frac{x_2 - x_1}{1 + x_2 x_1}, \quad (3.20)$$

where $y_{1,2}$ and $x_{1,2}$ are given by Equations (3.17) and (3.18), respectively.

In the case of free convective conditions, *i.e.*, $\zeta \ll 0$, scaling owing to Prandtl [53], Obukhov [54], and Priestley [55] is considered. It is based on the similarity hypothesis $F\left(z, H/(c_{p,0}\overline{\rho}), g/\Theta_m, \partial \hat{\Theta}/\partial z\right) = 0$ leading to

$$\frac{\partial \hat{\Theta}}{\partial z} = C \left(\frac{H}{c_{p,0}\overline{\rho}}\right)^{\frac{2}{3}} \left(\frac{g}{\Theta_m}\right)^{-\frac{1}{3}} z^{-\frac{4}{3}}, \quad (3.21)$$

where $C \cong -1.07$ is Priestley's constant (e.g., [13,34]). The negative sign is required to guarantee a lapse rate in case of free convective conditions for which $H > 0$ is considered. In accord with the definition of the Obukhov number (see Equation (2.16)), the free convective condition $\zeta \ll 0$ means that the friction velocity is of minor importance (e.g., [2,56]). It may be expressed by [2]

$$\zeta_{fc} = \lim_{u_* \to 0} \frac{z}{L} = -\lim_{u_* \to 0} \kappa \frac{g}{\Theta_m} \frac{H}{c_p \overline{\rho} u_*^3} z, \quad (3.22)$$

where the subscript "fc" indicates free convective conditions.

Integrating Equation (3.21) over the layer $[z_1, z_2]$ of the ASL leads to the Priestley-Estoque relation (e.g., [13, 57])

$$H = -\overline{\rho} c_{p,0} \Gamma_h \left(\hat{\Theta}_2 - \hat{\Theta}_1\right) \quad (3.23)$$

with

$$\Gamma_h = \frac{C_1}{3\left\{z_1^{-\frac{1}{3}} - z_2^{-\frac{1}{3}}\right\}} \left(-\frac{\frac{g}{\Theta_m}\left(\hat{\Theta}_2 - \hat{\Theta}_1\right)}{3\left\{z_1^{-\frac{1}{3}} - z_2^{-\frac{1}{3}}\right\}} \right)^{\frac{1}{2}}, \quad (3.24)$$

where $C_1 = |C|^{-3/2} \cong 0.90$. For $z_2 \to \infty$ one obtains the solution of Monin and Obukhov [2] presented by their Equation (61).

Priestley's constant was recently re-estimated by Dillon Amaya on the basis of data sets taken from the Hydrology-Atmosphere Pilot Experiment (HAPEX) in the Sahel of Niger, Africa. This field campaign was active from 1990-1992, but for the purpose of his re-estimation only data from September of 1992 were used. The corresponding data files were downloaded from the campaign's website (www.cesbio.ups-tlse.fr/hapex, retrieved 7/2/2012). The measurements were carried out over areas of degraded fallow bush, which had once been an agricultural center, but has been left to naturally restore its fertilization. Amaya found $C \cong -1.03$, *i.e.*, this value is, on average, 4% smaller than the commonly accepted value (for more details, see [58]).

Rearranging Equation (3.21) in the sense of Monin-Obukhov scaling, where only dry air is considered (*i.e.*, the influence of water vapor is ignored), leads to (e.g., [2,13,34,59])

$$\frac{\kappa(z-d)}{\Theta_*} \frac{\partial \hat{\Theta}}{\partial z} = -C \kappa^{4/3} (-\zeta)^{-1/3} \cong (-35.7\zeta)^{-1/3} \quad (3.25)$$

if Amaya's value for the Priestley constant and a von Karman constant of $\kappa = 0.40$ are assumed. For comparison: Based on the Cimljansk experiment, Zilitinkevič and Čalikov [43] found $\Phi_h(\zeta) = 0.40(-\zeta)^{-1/3} \cong (-15.6\zeta)^{-1/3}$, but only for the stability range $-1.2 < \zeta < -0.15$. Note that the local similarity functions for sensible heat given either by Equation (3.3) or Equation (3.7), commonly used, do not converge to the asymptotic solution (3.25) for $\zeta \ll 0$ (see **Figure 2**). The same is true in case of Equation (3.19). This means that the Businger-Pandolfo relationship $\Phi_h(\zeta) = \Phi_m^2(\zeta) \Leftrightarrow Ri = \zeta$ is not valid for the entire range of unstable stratification. To avoid this weakness, we suggest the following local similarity function for sensible heat [58]:

$$\Phi_h(\zeta) = (1 - \gamma_6 \zeta)^{-1/3} \quad (3.26)$$

with $\gamma_6 \cong 35.7$. It is illustrated in the lower part of **Figure 1** together with those deduced from various past field campaigns. Obviously, this local similarity function is in substantial agreement with Högström's [40] field data for the stability range $-2 < \zeta \leq 0$ and tends to the free convective conditions as described by Prandtl-Obukhov-

Priestley scaling (see **Figure 2**). The gradient Richardson number deduced on the basis of Equations (3.2), (3.4), and (3.26) is illustrated in **Figure 4**. As suggested by Dyer and Bradley [42] and Webb [47], the condition $|Ri| > |\zeta|$ is fulfilled, but the deviation from the one-to-one line characterized by $|Ri| = |\zeta|$ is notably stronger than empirically found, for instance, by Dyer and Bradley [42].

Equation (3.26) can be handled like Equation (3.2). Thus, one obtains:

$$\Psi_h(\zeta_2,\zeta_1) = \frac{3}{2}\ln\frac{y_2^2 + y_2 + 1}{y_1^2 + y_1 + 1}$$
$$-\sqrt{3}\arctan\frac{x_2 - x_1}{1 + x_2 x_1}, \tag{3.27}$$

$$y_{1,2} = \Phi_h^{-1}(\zeta_{1,2}) = (1 - \gamma_6\zeta_{1,2})^{1/3}, \tag{3.28}$$

and

$$x_{1,2} = (2y_{1,2} + 1)/\sqrt{3}. \tag{3.29}$$

The integral similarity functions given by Formulae (3.11), (3.20), and (3.27) are illustrated in **Figure 6**. Obviously, the difference between the formulae (3.11) and (3.27) is small, but both differ notably from that given by Equation (3.20).

4. Integral Similarity Functions for Stable Stratification

As mentioned before, Monin and Obukhov [2] already proposed for stable stratification (and weakly unstable stratification) a linear relationship of the form

$$\Phi_m(\zeta) = 1 + \gamma_7\zeta \tag{4.1}$$

later experimentally proved by Čalikov [60], Zilitinkevič
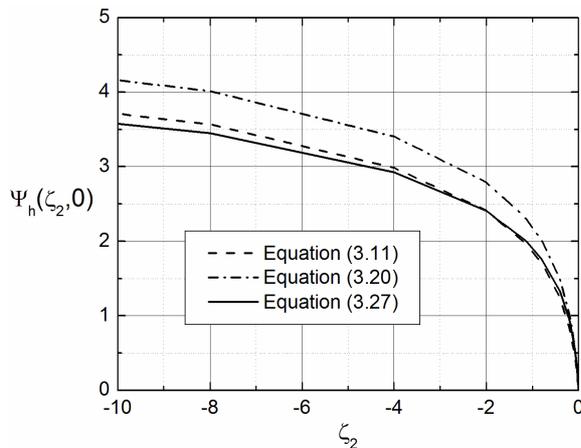
Figure 6. The integral similarity function for sensible heat, $\Psi_h(\zeta_2,0)$, provided by formulae (3.11), (3.20), and (3.27) and plotted against the Obukhov number $\zeta_2 = z_2/L$.

and Čalikov [43], Businger *et al.* [27] and others mainly for the stability range $0 \leq \zeta < 1$, but there is a large scatter in the case of momentum with some values of $\Phi_m(\zeta)$ for $\zeta > 1$ (see **Figures 7** and **8**). Dyer [31] and Panofsky and Dutton [28] recommended a value of $\gamma_7 \cong 5$ which is close to that of Webb [61]. This value is much larger than $\beta \cong 0.6$ mentioned before (see Equation (2.1)).

The integration of Formula (4.1) over the layer $[z_1, z_2]$ of the ASL provides for the integral similarity function

$$\Psi_m(\zeta_2,\zeta_1) = -\gamma_7(\zeta_2 - \zeta_1). \tag{4.2}$$
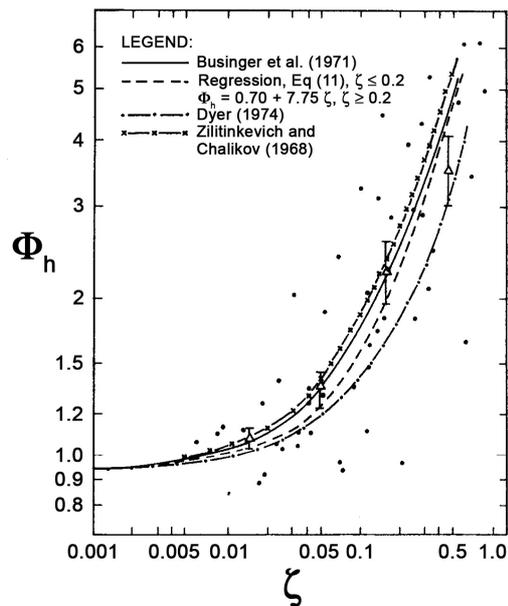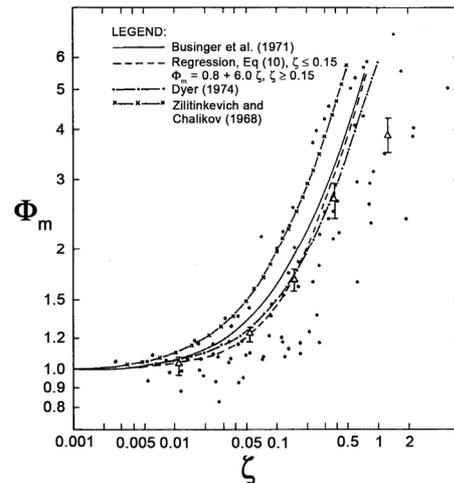
Figure 7. The local similarity functions $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ versus the Obukhov number ζ for stable stratification (adopted from Högström [40]). The dots represent Högström's field data. Note that the references Businger *et al.* (1971), Dyer (1974), and Zilitinkevich and Chalikov (1968) are listed here as references [27,31,43].
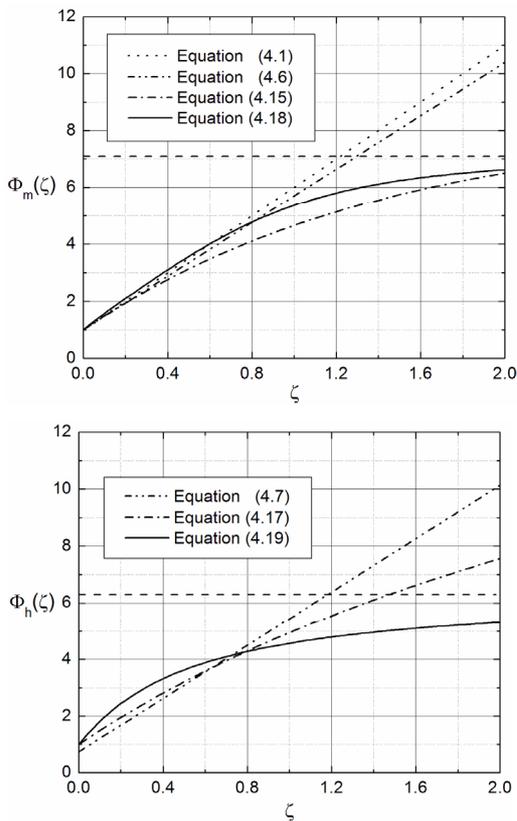
Figure 8. Local similarity functions $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ versus the Obukhov number $\zeta = z/L$ for stable stratification. The dashed lines characterize the asymptotic solutions of Cheng and Brutsaert [64] for $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$, respectively. Since Webb [61] assumed $\Phi_h(\zeta) = \Phi_m(\zeta)$, Equation (4.1) is omitted in the lower part of this figure. Furthermore, the formula recommended by Dyer [31] and illustrated in Figure 7 is nearly identical with that given by Equation (4.1). Moreover, in case of Equations (4.15) and (4.17) the constants of Beljaars and Holtslag [66] have been used.

This integral similarity function leads to a similar form as already suggested by Monin and Obukhov [2], even though the parameters β and γ_7 differ considerably. As aforementioned, Monin and Obukhov [2] also proposed the relationship

$$\Phi_h(\zeta) = \Phi_m(\zeta), \qquad (4.3)$$

also recommended by Webb [61]. In accord with Equation (3.4), this relationship provides

$$Ri = \frac{\zeta}{1 + \gamma_7 \zeta} \qquad (4.4)$$

or

$$\zeta = \frac{Ri}{1 - \gamma_7 Ri}. \qquad (4.5)$$

Based on the 1968 Kansas field experiment [48], Businger *et al.* [27], however, suggested for stable stratification:

$$\Phi_m(\zeta) = 1 + \gamma_8 \zeta \qquad (4.6)$$

and

$$\Phi_h(\zeta) = 0.74 + \gamma_8 \zeta \qquad (4.7)$$

with $\gamma_8 \cong 4.7$ (see **Figures 7** and **8**).

The integration of these formulae over the layer $[z_1, z_2]$ of the ASL yields

$$\Psi_m(\zeta_2, \zeta_1) = -\gamma_8(\zeta_2 - \zeta_1) \qquad (4.8)$$

and

$$\Psi_h(\zeta_2, \zeta_1) = -\gamma_8(\zeta_2 - \zeta_1) + 0.26 \ln \frac{\zeta_2}{\zeta_1}. \qquad (4.9)$$

As expected, the integral similarity functions (4.2) and (4.8) are nearly identical (see **Figure 9**). They only differ by the parameters γ_7 and γ_8. In contrast to this behavior, the Formula (4.9) notably differs from both other equations, where, in addition, ζ_1 must fulfill the condi-
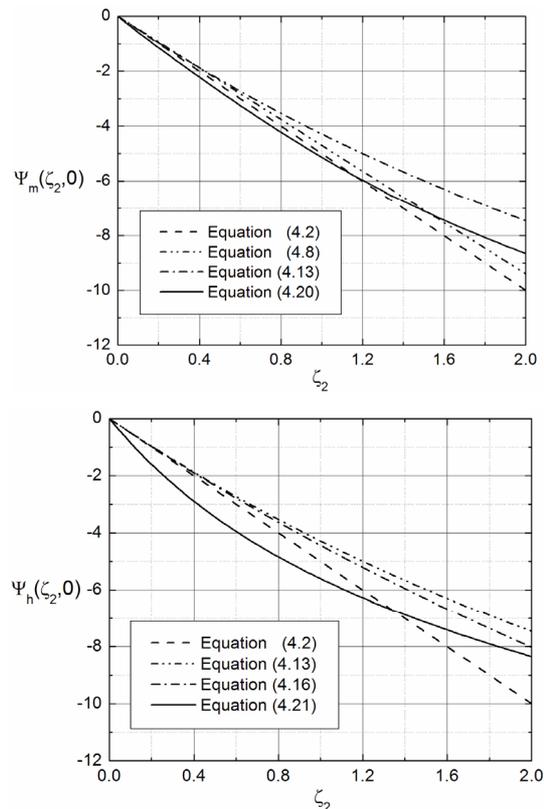
Figure 9. The integral similarity function for momentum and sensible heat, $\Psi_m(\zeta_2, 0)$ and $\Psi_h(\zeta_2, 0)$, versus the Obukhov number $\zeta_2 = z_2/L$ for stable stratification. Note that in case of Equations (4.13) and (4.16) the constants of Beljaars and Holtslag [66] have been used.

tion $\zeta_1 > 0$. Compared with Equation (2.11), it provides a notably different profile function

$$\hat{\Theta}(z_2) - \hat{\Theta}(z_1) = \frac{\Theta_*}{\kappa}\left(0.74\ln\frac{z_2}{z_1} - \gamma_8(\zeta_2 - \zeta_1)\right). \quad (4.10)$$

Inserting the local similarity functions (4.6) and (4.7) into Equation (3.4) provides

$$Ri = \frac{\zeta}{1+\gamma_8\zeta}\left(1 - \frac{0.26}{1+\gamma_8\zeta}\right). \quad (4.11)$$

Obviously, for stable stratification the influence of the term $0.26/(1+\gamma_{12}\zeta)$ weakens gradually as ζ increases, *i.e.*, as illustrated in **Figure 10**, the results inferred from Formulae (4.4) and (4.11) only differ slightly for strongly stable stratification. This small difference is mainly related to the parameters γ_7 and γ_8.

Two prominent difficulties can be attributed to the use of these parameterization principles:

Since $\gamma_7 \cong 5$ is commonly recommended, the gradient Richardson number has to satisfy the condition $Ri < 0.2$ because Equation (4.5) would become indeterminate for $Ri = 0.2$, and ζ would be negative for $Ri > 0.2$. The latter contradicts the definition of stable stratification for which $\zeta > 0$ [62]. Rearranging Equation (4.11) leads to

$$\zeta = +\sqrt{\frac{Ri}{\gamma_8(1-\gamma_8 Ri)} + \left(\frac{0.37 - \gamma_8 Ri}{\gamma_8(1-\gamma_8 Ri)}\right)^2} \\ - \frac{0.37 - \gamma_8 Ri}{\gamma_8(1-\gamma_8 Ri)}. \quad (4.12)$$

Thus, the condition $Ri < 0.2127$ has to be fulfilled to prevent that ζ becomes negative if $\gamma_8 \cong 4.7$ is used. This means that in these two cases the gradient Richardson number is always notably lower than the critical Richardson number customarily assumed to be
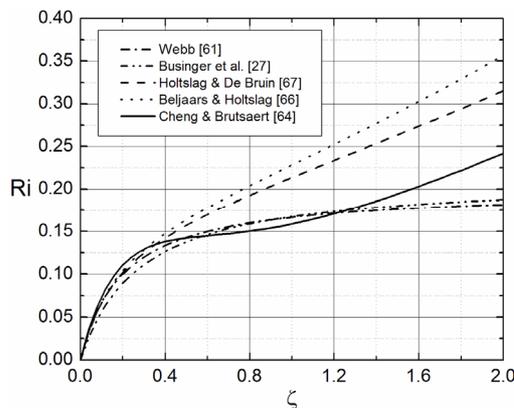
Figure 10. Gradient Richardson number, *Ri*, versus the Obukhov number ζ for stable stratification, where *Ri* has been deduced on the basis of Equation (3.4).

$Ri_{cr} \cong 0.25$ (e.g., [13,28]), even though Ellison [63] found gradient Richardson numbers up to $Ri = 1$ in wind tunnel experiments.

According to Högström [40] and Cheng and Brutsaert [64], reliable values of the Obukhov number satisfy the condition $0 < \zeta \le 2$ when stable stratification prevails. This means that for $\zeta = 2$ the gradient Richardson number amounts to $Ri = 0.182$ in case of Equation (4.4) and to $Ri = 0.188$ in case of Equation (4.11). As the gradient Richardson number and the flux Richardson number, Ri_f, are related to each other by $Ri = \Pr_t Ri_f$ and the turbulent Prandtl number is given by $\Pr_t = \Phi_h(\zeta)/\Phi_m(\zeta)$, the assumption $\Phi_h(\zeta) = \Phi_m(\zeta)$ [2,61] would lead to $\Pr_t = 1$, and, hence, to $Ri = Ri_f$. This means that under such condition the flux Richardson number would be restricted according to $Ri_f \le 0.182,\cdots,0.188$. This value is much smaller than the value of $Ri_f = 1$ that characterizes the fact that mechanical gain of TKE equals the thermal loss of TKE so that the turbulent flow becomes increasingly viscous (laminar) due to the dissipation of energy [13]. As discussed by Mölders and Kramm [62], the restriction of Ri was Louis' [65] reason to introduce a parametric model with which he artificially enhanced the transfer coefficient for sensible heat for strongly stable stratification to prevent "that once the bulk Richardson number (derived from Ri using finite differences) exceeds its critical value, the ground becomes energetically disconnected from the atmosphere and starts cooling by radiation at a faster rate than is actually observed".

To prevent such an energetic disconnection Beljaars and Holtslag [66] first discussed the following integral similarity functions for momentum and sensible heat that is based on the work of Holtslag and De Bruin [67]:

$$\Psi_m(\zeta_2,\zeta_1)$$
$$= -\gamma_9(\zeta_2 - \zeta_1) - \gamma_{10}\left\{\left(\zeta_2 - \frac{\gamma_{11}}{\gamma_{12}}\right)\exp(-\gamma_{12}\zeta_2)\right. \quad (4.13)$$
$$\left. - \left(\zeta_1 - \frac{\gamma_{11}}{\gamma_{12}}\right)\exp(-\gamma_{12}\zeta_1)\right\}$$

and $\Psi_h(\zeta) = \Psi_m(\zeta)$, where $\gamma_9 = 0.7$, $\gamma_{10} = 0.75$, $\gamma_{11} = 5$, and $\gamma_{12} = 0.35$. Here, this formula is presented for the layer $[z_1, z_2]$ of the ASL. For $\zeta_1 = 0$ one immediately obtains their original one (see **Figure 9**). Since

$$\frac{d\Psi_m(\zeta)}{d\zeta} = \frac{1-\Phi_m(\zeta)}{\zeta}, \quad (4.14)$$

the corresponding local similarity function reads [62]:

$$\Phi_m(\zeta) = 1 + \zeta\{\gamma_9 + \gamma_{10}\exp(-\gamma_{12}\zeta)(1+\gamma_{11}-\gamma_{12}\zeta)\}. \quad (4.15)$$

As illustrated in **Figure 8**, this formula notably differs

from Webb's [61] recommendation. The corresponding gradient Richardson number determined on the basis of Equation (3.4) is also shown in **Figure 10**. Obviously, these local similarity functions for momentum and sensible heat result in a gradient Richardson number up to $Ri \cong 0.32$ for $\zeta = 2$, *i.e.*, it is already larger than $Ri_{cr} \cong 0.25$. If ζ increases the Ri value will also increase. To obtain results more consistent with critical Richardson number considerations Beljaars and Holtslag [66] proposed for the integral similarity function for sensible heat:

$$\Psi_h\left(\zeta_2,\zeta_1\right)=\left(1+\frac{2}{3}\gamma_9\zeta_1\right)^{\frac{3}{2}}-\left(1+\frac{2}{3}\gamma_9\zeta_2\right)^{\frac{3}{2}}$$
$$-\gamma_{10}\left\{\left(\zeta_2-\frac{\gamma_{11}}{\gamma_{12}}\right)\exp\left(-\gamma_{12}\zeta_2\right)-\left(\zeta_1-\frac{\gamma_{11}}{\gamma_{12}}\right)\exp\left(-\gamma_{12}\zeta_1\right)\right\},$$
(4.16)

again presented here for the layer $\left[z_1, z_2\right]$ of the ASL. The corresponding the local similarity function reads:

$$\Phi_h\left(\zeta\right)=1+\gamma_9\zeta\left(1+\frac{2}{3}\gamma_9\zeta\right)^{\frac{3}{2}}$$
$$+\gamma_{10}\zeta\exp\left(-\gamma_{12}\zeta\right)\left(1+\gamma_{11}-\gamma_{12}\zeta\right).$$
(4.17)

Beljaars and Holtslag [66] recommended $\gamma_9 = 1.0$, $\gamma_{10} = 0.667$, $\gamma_{11} = 5$, and $\gamma_{12} = 0.35$ for both, Equation (4.13) and Equation (4.16). This change has the effect that the gradient Richardson number increases up to $Ri \cong 0.36$ for $\zeta = 2$, *i.e.*, it is still larger than $Ri_{cr} \cong 0.25$. Again, if ζ increases the Ri value will increase rapidly.

Recently, Cheng and Brutsaert [64] suggested for the entire range of stable stratification $\left(0 \le \zeta \le 2\right)$ following formulae:

$$\Phi_m\left(\zeta\right)=1+\gamma_{13}\left(\frac{\zeta+\zeta^{\gamma_{14}}\left(1+\zeta^{\gamma_{14}}\right)^{\frac{1-\gamma_{14}}{\gamma_{14}}}}{\zeta+\left(1+\zeta^{\gamma_{14}}\right)^{\frac{1}{\gamma_{14}}}}\right)$$
(4.18)

and

$$\Phi_h\left(\zeta\right)=1+\gamma_{15}\left(\frac{\zeta+\zeta^{\gamma_{16}}\left(1+\zeta^{\gamma_{16}}\right)^{\frac{1-\gamma_{16}}{\gamma_{16}}}}{\zeta+\left(1+\zeta^{\gamma_{16}}\right)^{\frac{1}{\gamma_{16}}}}\right),$$
(4.19)

where $\gamma_{13} = 6.1$, $\gamma_{14} = 2.5$, $\gamma_{15} = 5.3$, and $\gamma_{16} = 1.1$. For neutral conditions, *i.e.*, $\zeta = 0$, one obtains $\Phi_h\left(0\right) = \Phi_m\left(0\right) = 1$. For moderately stable stratification both formulae can be approximated by linear expressions, *i.e.*, $\Phi_m\left(\zeta\right) \cong 1+\gamma_{13}\zeta$ and $\Phi_h\left(\zeta\right) \cong 1+\gamma_{15}\zeta$, but for increasing stability formulae (4.18) and (4.19) tend to $\Phi_m\left(\zeta\right) = 1+\gamma_{13}$ and $\Phi_h\left(\zeta\right) = 1+\gamma_{15}$ (see **Figure 8**). Obviously, for the entire range of stable stratification

$\Phi_m\left(\zeta\right)$ and $\Phi_h\left(\zeta\right)$ differ from each other. In contrast to the functions $\Phi_m\left(\zeta\right)$ and $\Phi_h\left(\zeta\right)$ of Beljaars and Holtslag [66], which have points of inflection, the formulae of Cheng and Brutsaert [64] are bounded [62]. The relationship between the gradient Richardson number and the Obukhov number is also illustrated in **Figure 10**. Obviously, gradient Richardson numbers up to $Ri \cong 0.24$ occur for $\zeta \le 2$. As in case of the local similarity functions of Beljaars and Holtslag [66], if ζ increases the Ri value will also increase. Since the local similarity functions of Cheng and Brutsaert [64] are bounded, this increase of Ri is finally proportional to ζ.

The results for strongly stable stratification have been considered with care. As reported by Cheng and Brutsaert [64], the calculated $\Phi_h\left(\zeta\right)-1$ data points for $\zeta > 2$ were excluded from their analysis because the larger scatter suggested either unacceptable error in the measurements or perhaps other unexplained physical effects. According to them, a possible reason could be that these data points are already outside the stable surface layer so that Monin-Obukhov similarity for the transfer of momentum und sensible heat across the ASL may not further be valid [13].

The Formulae (4.18) and (4.19) provide logarithmic profiles for neutral and strongly stable stratification. The latter, already found by Webb [61] and Handorf *et al.* [68], seems to be awkward because if the magnitude of turbulent fluctuations decreases towards the small values of the quiet regime with increasing stability (e.g. [69,70]), the near-surface flow should become mainly laminar. For a pure laminar flow, viscous effects dominate leading to $\partial U/\partial\eta = u_*$, $\partial\hat{\Theta}/\partial\eta = \Pr\Theta_*$, and $\partial\hat{q}/\partial\eta = Sc_q q_*$, where $\Pr$ and Sc_q are the Prandtl number and the Schmidt number for water vapor, respectively. Thus, linear profiles have to be expected [13]. The same is true when the respective eddy diffusivities become invariant with height. Such height invariance might be possible when the quiet regime prevails and the magnitude of the turbulent fluctuations is small across the entire ASL. Thus, we have to assume that Monin-Obukhov similarity is incomplete under strongly stable conditions. If under such conditions the constant flux approximation is no longer valid as debated, for instance, by Webb [61] and Poulos and Burns [71], Monin-Obukhov similarity must not be expected [13,62].

The solutions for the local similarity functions $\Phi_m\left(\zeta\right)$ and $\Phi_h\left(\zeta\right)$ given by Formulae (4.18) and (4.19) for the layer $\left[z_1, z_2\right]$ of the ASL read:

$$\Psi_m\left(\zeta_2,\zeta_1\right)=-\gamma_{13}\ln\frac{\zeta_2+\left(1+\zeta_2^{\gamma_{14}}\right)^{\frac{1}{\gamma_{14}}}}{\zeta_1+\left(1+\zeta_1^{\gamma_{14}}\right)^{\frac{1}{\gamma_{14}}}}$$
(4.20)

and

$$\Psi_h\left(\zeta_2,\zeta_1\right) = -\gamma_{15}\ln\frac{\zeta_2 + \left(1+\zeta_2^{\gamma_{16}}\right)^{\frac{1}{\gamma_{16}}}}{\zeta_1 + \left(1+\zeta_1^{\gamma_{16}}\right)^{\frac{1}{\gamma_{16}}}}. \qquad (4.21)$$

For $\zeta_1 = 0$, one obtains the original expressions of Cheng and Brutsaert [64]. These integral similarity functions are illustrated in **Figure 9**, together with those related to Webb [61], Businger *et al.* [27], and Beljaars and Holtslag [66].

5. Final Remarks and Conclusions

With his new profile function for the mean, horizontal wind speed and his integral similarity function for momentum HAP opened the door for Monin-Obukhov scaling in a wide range of micrometeorological and microclimatological applications. All empirical and semitheoretical expressions for the local similarity functions $\Phi_m\left(\zeta\right)$, $\Phi_h\left(\zeta\right)$, and $\Phi_q\left(\zeta\right)$ that can be found in the literature may be inserted in his concept of the integral similarity function, even though numerical integration might be required. Fortunately, in most cases the integration can be performed elementarily. For the aforementioned applications, we currently recommend Equations (3.2) and (3.26) for unstable stratification $\left(\zeta < 0\right)$, leading to the integral similarity functions (3.16) and (3.27), and Equations (4.18) and (4.19) for stable stratification $\left(0 \leq \zeta \leq 2\right)$, leading to the integral similarity functions (4.20) and (4.21).

As there are some indications that Monin-Obukhov similarity is no longer valid in case of strongly stable stratification $\left(\zeta > 2\right)$ probably owing to incomplete similarity, further research should focus on this range of diabatic stratification.

6. Acknowledgements

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