Distributive-Law Semantics for Cellular Automata and Agent-Based Models
Appendix

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This is the appendix and electronic supplement to the article titled as above, appearing in the Proceedings of the 4th Conference on Algebra and Coalgebra (CALCO 2011), Lecture Notes in Computer Science, Springer-Verlag.

Appendix

A Omitted Proofs and Examples

Proof (Lemma 2). Let (A, α) be an initial Σ-algebra. The unique extension to a λ-bialgebra, by clause (b) of Definition 5, is $(B^\lambda \alpha)?$. It remains to show that the λ-bialgebra $\big(A, \alpha, (B^\lambda \alpha)?\big)$ is initial: Let (X, f, g) be any λ-bialgebra. Then $f? : \alpha \to f$ is the unique Σ-algebra homomorphism. We show that it is also a B-coalgebra homomorphism $f? : (B^\lambda \alpha)? \to g$, that is $g \circ f? = Bf? \circ (B^\lambda \alpha)?$ (see Fig. 8, dotted arrows), by demonstrating that both sides of the equation are Σ-algebra homomorphisms $\alpha \to B^\lambda f$, of which there is only one, namely $(B^\lambda f)?$.

1. $g \circ f? : \alpha \to B^\lambda f$ is just the composition of $f? : \alpha \to f$ and $g : f \to B^\lambda f$.
2. Note that functor B takes the commuting diagram for $f? : \alpha \to f$ to another commuting diagram (see Fig. 8, bottom) that is used in step (†) below:

$$\begin{aligned}
& B^\lambda f \circ \Sigma\big(Bf? \circ (B^\lambda \alpha)?\big) \\
\text{(definition)} &= Bf \circ \lambda_X \circ \Sigma\big(Bf? \circ (B^\lambda \alpha)?\big) \\
\text{(functor } \Sigma) &= Bf \circ \lambda_X \circ \Sigma Bf? \circ \Sigma(B^\lambda \alpha)? \\
\text{(natural } \lambda) &= Bf \circ B\Sigma f? \circ \lambda_A \circ \Sigma(B^\lambda \alpha)? \\
(\dagger) &= Bf? \circ B\alpha \circ \lambda_A \circ \Sigma(B^\lambda \alpha)? \\
\text{(bialgebra)} &= Bf? \circ (B^\lambda \alpha)? \circ \alpha
\end{aligned}$$

Hence $Bf? \circ (B^\lambda \alpha)? : \alpha \to f$. □

Proof (Theorem 2). Naturality follows from naturality of sl^+. Verify context axioms:

(C1) Shape preservation of γ follows from shape preservation of $\widehat{\chi}$:

$$\begin{aligned}
W\dagger\big(\gamma_1(c \vdash x)\big) &= W\dagger\Big(WC^\sharp\big(sl^+(c \vdash x)\big)\big(\widehat{\chi}_1(x)\big)\Big) \\
&= \Big(W\dagger \circ WC^\sharp\big(sl^+(c \vdash x)\big)\Big)\big(\widehat{\chi}_1(x)\big) \\
\text{(functor } W) &= W\Big(\underbrace{\dagger \circ C^\sharp\big(sl^+(c \vdash x)\big)}_{: C^\sharp \mathbb{Z}^2 \to 1}\Big)\big(\widehat{\chi}_1(x)\big) \\
\text{(uniqueness)} &= W\ddagger\big(\widehat{\chi}_1(x)\big) \\
\text{(Lemma 5)} &= x
\end{aligned}$$

(C2) Since sl^+ is defined in terms of $sl^\star$ only, γ cannot distinguish similar neighborhoods. □

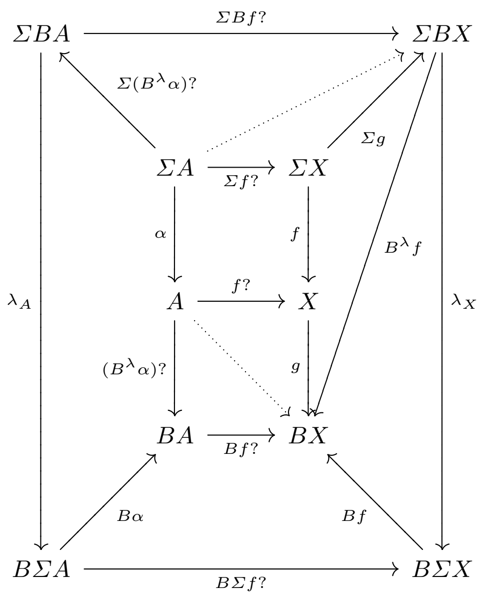

Fig. 8. Proof diagram for Lemma 2.

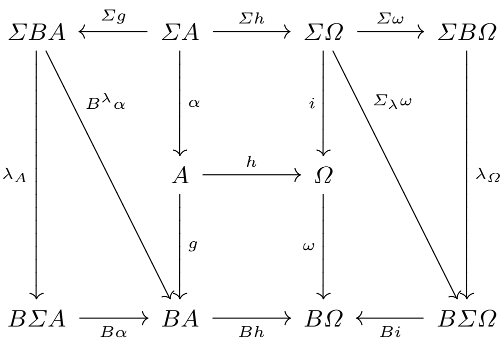

Fig. 9. Proof diagram for Theorem 1. Morphisms are unique: $g = (B^{\lambda}\alpha)?$ by initiality of α and $i = (\Sigma_{\lambda}\omega)!$ by finality of ω, respectively; hence $h = (B^{\lambda}\alpha)?! = (\Sigma_{\lambda}\omega)!?$.

Example 5. Compositionality laws for the von Neumann context as specified in Example 4 can be given as:

$$\begin{aligned}
\mathit{cosingleton}_L\big((n,e,s,w)\big) &= (\swarrow n, \nwarrow e, \nwarrow s, \nearrow w) \\
\mathit{cohwrap}_L\big(x,(n,e,s,w)\big) &= (n,x,s,x) \\
\mathit{covwrap}_L\big(x,(n,e,s,w)\big) &= (x,e,x,w) \\
\mathit{cobeside}_L\big(x_1,x_2,(n,e,s,w)\big) &= \big((n_1,x_2,s_1,w),(n_2,e,s_2,x_1)\big) \\
\mathit{coabove}_L\big(x_1,x_2,(n,e,s,w)\big) &= \big((n,e_1,x_2,w_1),(x_1,e_2,s,w_2)\big)
\end{aligned}$$

$$(n_1,n_2) = \mathit{hsplit}\big(n, \mathit{wd}(x_1)\big) \qquad (e_1,e_2) = \mathit{vsplit}\big(e, \mathit{ht}(x_1)\big)$$

$$(s_1,s_2) = \mathit{hsplit}\big(s, \mathit{wd}(x_1)\big) \qquad (w_1,w_2) = \mathit{vsplit}\big(w, \mathit{ht}(x_1)\big)$$

where

1. the diagonal arrows select corner elements: $\swarrow x = \mathit{sl}^\star(x)(\mathit{ht}(x)-1,0)$ etc.
2. the operation *hsplit* maps (x,k) to some pair (x_1,x_2) such that $\mathit{wd}(x_1)=k$ and $x_1 \mid x_2 \approx x$.
3. the operation *vsplit* maps (x,k) to some pair (x_1,x_2) such that $\mathit{ht}(x_1)=k$ and $x_1 \mathbin{/} x_2 \approx x$.

Lemma 6. *The laws given above are compatible with the distributive law of the von Neumann context as specified in Example 4.*

Proof. Straightforward, but rather verbose syntactic case dinstinction. Let $c = (n,e,s,w)$.

$$\begin{aligned}
W^\gamma u(c \vdash x) &= (Wu \circ \gamma_L)(c \vdash x) \\
&= Wu\big(\gamma_L(c \vdash x)\big) \\
&= Wu\Big(WC^\sharp\big(\mathit{sl}^+(c \vdash x)\big)\big(\widehat{\chi}_L(x)\big)\Big)
\end{aligned}$$

Now abbreviate $\mathit{sl}^+(c \vdash x)$ to v.

1. Singleton:

$$\begin{aligned}
W^\gamma u\big(c \vdash [a]\big) &= Wu\Big(WC^\sharp v\Big(\widehat{\chi}_L\big([a]\big)\Big)\Big) \\
&= Wu\Big(WC^\sharp v\Big(\big[\chi \vdash (0,0)\big]\Big)\Big) \\
&= Wu\Big(WC^\sharp v\Big(\big[(-1,0),(0,1),(1,0),(0,-1) \vdash (0,0)\big]\Big)\Big) \\
&= \Big[u\Big(C^\sharp v\big((-1,0),(0,1),(1,0),(0,-1) \vdash (0,0)\big)\Big)\Big] \\
&= \Big[u\big(v(-1,0), v(0,1), v(1,0), v(0,-1) \vdash v(0,0)\big)\Big] \\
&= \big[u(\swarrow n, \nwarrow e, \nwarrow s, \nearrow w \vdash a)\big] \\
&= \big[u(\mathit{cosingleton}_L(c) \vdash a)\big]
\end{aligned}$$

2. Wrapping (only $\leftrightarrow$ shown, $\updownarrow$ dual):

$$
\begin{aligned}
W^{\gamma}u(c \vdash x^{\leftrightarrow}) &= Wu\Big(WC^{\sharp}v\big(\widehat{\chi}_L(x^{\leftrightarrow})\big)\Big) \\
&= Wu\bigg(WC^{\sharp}v\Big(\big(WC^{\sharp}f_x(\widehat{\chi}_L(x))\big)^{\leftrightarrow}\Big)\bigg) \\
&= \Big(Wu\big(WC^{\sharp}(v \circ f_x)(\widehat{\chi}_L(x))\big)\Big)^{\leftrightarrow}
\end{aligned}
$$

$$
\big(W^{\gamma}u(\mathit{cohwrap}_L(x,c) \vdash x)\big)^{\leftrightarrow} = \Big(Wu\Big(WC^{\sharp}\big(sl^{+}(\mathit{cohwrap}_L(x,c) \vdash x)\big)\big(\widehat{\chi}_L(x)\big)\Big)\Big)^{\leftrightarrow}
$$

Verify that $v \circ f_x = sl^{+}(c \vdash x) \circ f_x = sl^{+}(\mathit{cohwrap}_L(x,c) \vdash x)$ by case distinction according to the definition of sl^{+}.

3. Composition (only | shown, / dual):

$$
\begin{aligned}
W^{\gamma}u(c \vdash x_1 \mid x_2) &= Wu\Big(WC^{\sharp}v\big(\widehat{\chi}_L(x_1 \mid x_2)\big)\Big) \\
&= Wu\Big(WC^{\sharp}v\Big(\widehat{\chi}_L(x_1) \mid WC^{\sharp}h_x\big(\widehat{\chi}_L(x_2)\big)\Big)\Big) \\
&= Wu\Big(WC^{\sharp}v\big(\widehat{\chi}_L(x_1)\big)\Big) \mid Wu\Big(WC^{\sharp}(v \circ h_x)\big(\widehat{\chi}_L(x_2)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
W^{\gamma}u(c_1 \vdash x_1) \mid W^{\gamma}u(c_2 \vdash x_2) = \; & Wu\Big(WC^{\sharp}\big(sl^{+}(c_1 \vdash x_1)\big)\big(\widehat{\chi}_L(x_1)\big)\Big) \\
& \Big| \; Wu\Big(WC^{\sharp}\big(sl^{+}(c_2 \vdash x_2)\big)\big(\widehat{\chi}_L(x_2)\big)\Big)
\end{aligned}
$$

Verify that $(c_1, c_2) = \mathit{cobeside}_L(x_1, x_2, c)$ implies $sl^{+}(c_1 \vdash x_1) = sl^{+}(c \vdash x)$ and $sl^{+}(c_2 \vdash x_2) = sl^{+}(c \vdash x) \circ h_x$, on the domains $ht(x1) \times wd(x_1)$ and $ht(x_2) \times wd(x_2)$, respectively. □

Proof (Theorem 3). Show that $\lambda^u_Y \circ \Sigma_L B^{\mathcal{C}}_{WL} f = B^{\mathcal{C}}_{WL} \Sigma_L f \circ \lambda^u_X$ for all $f : X \to Y$. By completeness, for each $r \in \Sigma_L B^{\mathcal{C}}_{WL} X$ there is a rule such that

$$
\begin{aligned}
\lambda^u_Y\big(\Sigma_L B^{\mathcal{C}}_{WL} f(r)\big) &= \lambda^u_Y\Big(\Sigma_L B^{\mathcal{C}}_{WL} f\big(k(s_1 \rhd t_1, \ldots, s_n \rhd t_n)\big)\Big) \\
&= \lambda^u_Y\Big(k\big(B^{\mathcal{C}}_{WL} f(s_1 \rhd t_1), \ldots, B^{\mathcal{C}}_{WL} f(s_1 \rhd t_1)\big)\Big) \\
&= \lambda^u_Y\big(k(s_1 \rhd f \circ t_1, \ldots, s_1 \rhd f \circ t_1)\big) \\
&= s \rhd l_Y \circ (f \circ t_1 \times \cdots \times f \circ t_n) \circ d(s_1, \ldots, s_n, \_) \\
&= s \rhd l_Y \circ f^n \circ (t_1 \times \cdots \times t_n) \circ d(s_1, \ldots, s_n, \_) \\
\text{(by syntactic naturality)} &= s \rhd \Sigma_L f \circ l_X \circ (t_1 \times \cdots \times t_n) \circ d(s_1, \ldots, s_n, \_) \\
&= B^{\mathcal{C}}_{WL} \Sigma_L f\big(s \rhd l_X \circ (t_1 \times \cdots \times t_n) \circ d(s_1, \ldots, s_n, \_)\big) \\
&= B^{\mathcal{C}}_{WL} \Sigma_L f\Big(\lambda^u_X\big(k(s_1 \rhd t_1, \ldots, s_n \rhd t_n)\big)\Big) \\
&= B^{\mathcal{C}}_{WL} \Sigma_L f\big(\lambda^u_X(r)\big)
\end{aligned}
$$

□

Proof (Fragment of Theorem 4). Show that $\big(A, \alpha, (W^\gamma u)^{\triangleright}\big)$ is a λ^u-bialgebra.

$$\begin{aligned}
(W^\gamma u)^{\triangleright} \circ \alpha &= B\alpha \circ \lambda^u_A \circ \Sigma (W^\gamma u)^{\triangleright} \\
(W^\gamma u)^{\triangleright}\big(\alpha(s)\big) &= B\alpha\Big(\lambda^u_A\big(\Sigma(W^\gamma u)^{\triangleright}(s)\big)\Big) \\
\alpha(s) \triangleright W^\gamma u\big(_ \vdash \alpha(s)\big) &= B\alpha(s' \triangleright t) \qquad\qquad s' \triangleright t = \lambda^u_A\big(\Sigma(W^\gamma u)^{\triangleright}(s)\big) \\
&= s' \triangleright \alpha \circ t
\end{aligned}$$

Proceed by syntactic case distinction:

1. $\alpha(s) = [a]$.

$$\begin{aligned}
s' \triangleright t &= \lambda^u_A\Big(\Sigma(W^\gamma u)^{\triangleright}\big([a]\big)\Big) \\
&= \lambda^u_A\big([a]\big) \\
\text{by (2)} &= [a] \triangleright \mathit{singleton}_A \circ u(_ \vdash a) \circ \mathit{cosingleton}_L \\
\text{by (1)} &= [a] \triangleright \alpha^{-1} \circ W^\gamma u\big(_ \vdash [a]\big)
\end{aligned}$$

2. $\alpha(s) = s_1^{\leftrightarrow}$.

$$\begin{aligned}
s' \triangleright t &= \lambda^u_A\big(\Sigma(W^\gamma u)^{\triangleright}(s_1^{\leftrightarrow})\big) \\
&= \lambda^u_A\big((W^\gamma u)^{\triangleright}(s_1)^{\leftrightarrow}\big) \\
&= \lambda^u_A\Big(\big(s_1 \triangleright W^\gamma u(_ \vdash s_1)\big)^{\leftrightarrow}\Big) \\
\text{by (2)} &= s_1^{\leftrightarrow} \triangleright \mathit{hwrap}_A \circ W^\gamma u(_ \vdash s_1) \circ \mathit{cohwrap}_L(s_1, _) \\
\text{by (1)} &= s_1^{\leftrightarrow} \triangleright \alpha^{-1} \circ W^\gamma u(_ \vdash s_1^{\leftrightarrow})
\end{aligned}$$

3. $\alpha(s) = s_1^{\updownarrow}$ dual.
4. $\alpha(s) = s_1 \mid s_2$.

$$\begin{aligned}
s' \triangleright t &= \lambda^u_A\big(\Sigma(W^\gamma u)^{\triangleright}(s_1 \mid s_2)\big) \\
&= \lambda^u_A\big((W^\gamma u)^{\triangleright}(s_1) \mid (W^\gamma u)^{\triangleright}(s_2)\big) \\
&= \lambda^u_A\Big(\big(s_1 \triangleright W^\gamma u(_ \vdash s_1)\big) \mid \big(s_2 \triangleright W^\gamma u(_ \vdash s_2)\big)\Big) \\
\text{by (2)} &= s_1 \mid s_2 \triangleright \mathit{beside}_A \circ \big(W^\gamma u(_ \vdash s_1) \times W^\gamma u(_ \vdash s_2)\big) \circ \mathit{cobeside}_L(s_1, s_2, _) \\
\text{by (1)} &= s_1 \mid s_2 \triangleright \alpha^{-1} \circ W^\gamma u(_ \vdash s_1 \mid s_2)
\end{aligned}$$

5. $\alpha(s) = s_1 \,/\, s_2$ dual.

□

Example 6 (Worked-out Semantics). Consider a world similar to the one depicted in Fig. 4 and 5, left hand side, but with a nontrivial topology due to nested wrapping; namely of the shape

$$x = \big([\mathsf{I}] \mid [\mathsf{J}] \;/\; [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big)^{\leftrightarrow}$$

for $\mathsf{I}, \mathsf{J}, \mathsf{K}, \mathsf{L} \in L$. We use the geometrical orientation of the von Neumann context, writing

$$\left(\begin{smallmatrix} & n & \\ w & x & e \\ & s & \end{smallmatrix}\right)$$

for the template $(n, e, s, w \vdash x)$. The relocation map for singleton elements is (cf. Example 4)

$$\widehat{\chi}_L\big([a]\big) = \left(\begin{smallmatrix} & (-1,0) & \\ (0,-1) & (0,0) & (0,+1) \\ & (+1,0) & \end{smallmatrix}\right)$$

From this, the relocation map of our world is calculated bottom-up as

$$\begin{aligned}
\widehat{\chi}_L\big([\mathsf{L}]^{\updownarrow}\big) &= \left(\begin{smallmatrix} & (0,0) & \\ (0,-1) & (0,0) & (0,+1) \\ & (0,0) & \end{smallmatrix}\right)^{\updownarrow} \\
\widehat{\chi}_L\big([\mathsf{I}] \mid [\mathsf{J}]\big) &= \left(\begin{smallmatrix} & (-1,0) & \\ (0,-1) & (0,0) & (0,+1) \\ & (+1,0) & \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} & (-1,+1) & \\ (0,0) & (0,+1) & (0,+2) \\ & (+1,+1) & \end{smallmatrix}\right) \\
\widehat{\chi}_L\big([\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big) &= \left(\begin{smallmatrix} & (-1,0) & \\ (0,-1) & (0,0) & (0,+1) \\ & (+1,0) & \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} & (0,+1) & \\ (0,0) & (0,+1) & (0,+2) \\ & (0,+1) & \end{smallmatrix}\right)^{\updownarrow}
\end{aligned}$$

$$\widehat{\chi}_L\big([\mathsf{I}] \mid [\mathsf{J}] \;/\; [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big) = \begin{array}{l} \left(\begin{smallmatrix} & (-1,0) & \\ (0,-1) & (0,0) & (0,+1) \\ & (+1,0) & \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} & (-1,+1) & \\ (0,0) & (0,+1) & (0,+2) \\ & (+1,+1) & \end{smallmatrix}\right) \\ / \left(\begin{smallmatrix} & (0,0) & \\ (+1,-1) & (+1,0) & (+1,+1) \\ & (+2,0) & \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} & (+1,+1) & \\ (+1,0) & (+1,+1) & (+1,+2) \\ & (+1,+1) & \end{smallmatrix}\right)^{\updownarrow} \end{array}$$

$$\widehat{\chi}_L(x) = \left(\begin{array}{l} \left(\begin{smallmatrix} & (-1,0) & \\ (0,+1) & (0,0) & (0,+1) \\ & (+1,0) & \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} & (-1,+1) & \\ (0,0) & (0,+1) & (0,0) \\ & (+1,+1) & \end{smallmatrix}\right) \\ / \left(\begin{smallmatrix} & (0,0) & \\ (+1,+1) & (+1,0) & (+1,+1) \\ & (+2,0) & \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} & (+1,+1) & \\ (+1,0) & (+1,+1) & (+1,0) \\ & (+1,+1) & \end{smallmatrix}\right) \end{array}\right)^{\leftrightarrow}$$

The extended selection operation for von Neumann neighborhoods (Example 4) for our world yields

$$sl^{+}(c \vdash x) = \left\{\begin{array}{cccc} & (-1,0) \mapsto \mathsf{c} & (-1,+1) \mapsto \mathsf{d} & \\ (0,-1) \mapsto \mathsf{f} & (0,0) \mapsto \mathsf{I} & (0,+1) \mapsto \mathsf{J} & (0,+2) \mapsto \mathsf{m} \\ (+1,-1) \mapsto \mathsf{h} & (+1,0) \mapsto \mathsf{K} & (+1,+1) \mapsto \mathsf{L} & (+1,+2) \mapsto \mathsf{o} \\ & (+2,0) \mapsto \mathsf{q} & (+2,+1) \mapsto \mathsf{r} & \end{array}\right\}$$

where $c = (n, e, s, w)$ and

$$\begin{array}{llll} \mathsf{c} = sl^{\star}(n)\big(ht(n)-1,0\big) & \mathsf{m} = sl^{\star}(e)(0,0) & \mathsf{q} = sl^{\star}(s)(0,0) & \mathsf{f} = sl^{\star}(w)\big(0,wd(w)-1\big) \\ \mathsf{d} = sl^{\star}(n)\big(ht(n)-1,1\big) & \mathsf{o} = sl^{\star}(e)(1,0) & \mathsf{r} = sl^{\star}(s)(0,1) & \mathsf{h} = sl^{\star}(w)\big(1,wd(w)-1\big) \end{array}$$

Relocation map and extended selection define a distributive law; applied to our world we find

$$
\begin{aligned}
\gamma_L(c \vdash x) &= WC^\sharp\big(sl^+(c \vdash x)\big)\big(\widehat{\chi}_L(x)\big) \\
&= \left(\left[\left(\begin{smallmatrix} & \mathsf{c} & \\ \mathsf{J} & \mathsf{I} & \mathsf{J} \\ & \mathsf{K} & \end{smallmatrix}\right)\right] \mid \left[\left(\begin{smallmatrix} & \mathsf{d} & \\ \mathsf{I} & \mathsf{J} & \mathsf{I} \\ & \mathsf{L} & \end{smallmatrix}\right)\right] \mathbin{/} \left[\left(\begin{smallmatrix} & \mathsf{I} & \\ \mathsf{L} & \mathsf{K} & \mathsf{L} \\ & \mathsf{q} & \end{smallmatrix}\right)\right] \mid \left[\left(\begin{smallmatrix} & \mathsf{L} & \\ \mathsf{K} & \mathsf{L} & \mathsf{K} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]^{\updownarrow}\right)^{\leftrightarrow}
\end{aligned}
$$

and consequently a globalized update

$$
\begin{aligned}
W^\gamma u(c \vdash x) &= (Wu \circ \gamma_L)(c \vdash x) \\
&= \left(\left[u\left(\begin{smallmatrix} & \mathsf{c} & \\ \mathsf{J} & \mathsf{I} & \mathsf{J} \\ & \mathsf{K} & \end{smallmatrix}\right)\right] \mid \left[u\left(\begin{smallmatrix} & \mathsf{d} & \\ \mathsf{I} & \mathsf{J} & \mathsf{I} \\ & \mathsf{L} & \end{smallmatrix}\right)\right] \mathbin{/} \left[u\left(\begin{smallmatrix} & \mathsf{I} & \\ \mathsf{L} & \mathsf{K} & \mathsf{L} \\ & \mathsf{q} & \end{smallmatrix}\right)\right] \mid \left[u\left(\begin{smallmatrix} & \mathsf{L} & \\ \mathsf{K} & \mathsf{L} & \mathsf{K} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]^{\updownarrow}\right)^{\leftrightarrow}
\end{aligned}
$$

and automaton operation

$$(W^\gamma u)^{\triangleright}(x) = x \triangleright W^\gamma u(\_ \vdash x)$$

On the other hand, the divide-and-conquer strategy requires both top-down and bottom-up calculation. In general:

$$
\begin{aligned}
(B^{\lambda^u}\alpha)? \circ \alpha &= B\alpha \circ \lambda^u_A \circ \Sigma(B^{\lambda^u}\alpha)? \\
(B^{\lambda^u}\alpha)? &= B\alpha \circ \lambda^u_A \circ \Sigma(B^{\lambda^u}\alpha)? \circ \alpha^{-1}
\end{aligned}
$$

Syntax-related top-down phase:

$$
\begin{aligned}
s_0 \triangleright t_0 = (B^{\lambda^u}\alpha)?(x) &= (B\alpha \circ \lambda^u_A \circ \Sigma(B^{\lambda^u}\alpha)? \circ \alpha^{-1})(x) \\
&= (B\alpha \circ \lambda^u_A \circ \Sigma(B^{\lambda^u}\alpha)?)\Big(\big([\mathsf{I}] \mid [\mathsf{J}] \mathbin{/} [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big)^{\leftrightarrow}\Big) \\
&= (B\alpha \circ \lambda^u_A)\Big(\big((B^{\lambda^u}\alpha)?\big([\mathsf{I}] \mid [\mathsf{J}] \mathbin{/} [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big)\big)^{\leftrightarrow}\Big) \\
&= B\alpha\big(s_1^{\leftrightarrow} \triangleright \mathit{hwrap}_{WL} \circ t_1 \circ \mathit{cohwrap}_L(s_1, \_)\big) \\
&\qquad \text{where } s_1 \triangleright t_1 = (B^{\lambda^u}\alpha)?\big([\mathsf{I}] \mid [\mathsf{J}] \mathbin{/} [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big) \\
&= \big(\mathrm{id}_{WL} \times (\alpha \circ \_)\big)\big(s_1^{\leftrightarrow} \triangleright \mathit{hwrap}_{WL} \circ t_1 \circ \mathit{cohwrap}_L(s_1, \_)\big) \\
&= s_1^{\leftrightarrow} \triangleright \alpha \circ \mathit{hwrap}_{WL} \circ t_1 \circ \mathit{cohwrap}_L(s_1, \_)
\end{aligned}
$$

$$
\begin{aligned}
(B^{\lambda^u}\alpha)?\big([\mathsf{I}] \mid [\mathsf{J}] \mathbin{/} [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big) &= (B\alpha \circ \lambda^u_A)\Big((B^{\lambda^u}\alpha)?\big([\mathsf{I}] \mid [\mathsf{J}]\big) \mathbin{/} (B^{\lambda^u}\alpha)?\big([\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big)\Big) \\
&= s_2 \mathbin{/} s_3 \triangleright \alpha \circ \mathit{above}_{WL} \circ (t_2 \times t_3) \circ \mathit{coabove}_L(s_2, s_3, \_) \\
&\qquad \text{where } \begin{cases} s_2 \triangleright t_2 = (B^{\lambda^u}\alpha)?\big([\mathsf{I}] \mid [\mathsf{J}]\big) \\ s_3 \triangleright t_3 = (B^{\lambda^u}\alpha)?\big([\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big) \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(B^{\lambda^u}\alpha)?\big([\mathsf{I}] \mid [\mathsf{J}]\big) &= (B\alpha \circ \lambda^u_A)\Big((B^{\lambda^u}\alpha)?\big([\mathsf{I}]\big) \mid (B^{\lambda^u}\alpha)?\big([\mathsf{J}]\big)\Big) \\
&= s_4 \mid s_5 \triangleright \alpha \circ \mathit{beside}_{WL} \circ (t_4 \times t_5) \circ \mathit{beside}_L(s_4, s_5, \_) \\
&\qquad \text{where } \begin{cases} s_4 \triangleright t_4 = (B^{\lambda^u}\alpha)?\big([\mathsf{I}]\big) \\ s_5 \triangleright t_5 = (B^{\lambda^u}\alpha)?\big([\mathsf{J}]\big) \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(B^{\lambda^u}\alpha)?\big([\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\big) &= (B\alpha \circ \lambda^u_A)\Big((B^{\lambda^u}\alpha)?\big([\mathsf{K}]\big) \mid (B^{\lambda^u}\alpha)?\big([\mathsf{L}]^{\updownarrow}\big)\Big)\\
&= s_6 \mid s_7 \triangleright \alpha \circ \mathit{beside}_{WL} \circ (t_6 \times t_7) \circ \mathit{beside}_L(s_6, s_7, \_)\\
&\qquad \text{where } \begin{cases} s_6 \triangleright t_6 = (B^{\lambda^u}\alpha)?\big([\mathsf{K}]\big)\\ s_7 \triangleright t_7 = (B^{\lambda^u}\alpha)?\big([\mathsf{L}]^{\updownarrow}\big)\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(B^{\lambda^u}\alpha)?\big([\mathsf{L}]^{\updownarrow}\big) &= (B\alpha \circ \lambda^u_A)\bigg(\Big((B^{\lambda^u}\alpha)?\big([\mathsf{L}]\big)\Big)^{\updownarrow}\bigg)\\
&= s_8^{\updownarrow} \triangleright \alpha \circ \mathit{hwrap}_{WL} \circ t_8 \circ \mathit{covwrap}_L(s_8, \_)\\
&\qquad \text{where } s_8 \triangleright t_8 = (B^{\lambda^u}\alpha)?\big([\mathsf{L}]\big)
\end{aligned}
$$

Singletons:

$$
\begin{aligned}
(B^{\lambda^u}\alpha)?\big([a]\big) &= (B\alpha \circ \lambda^u_A)\big([a]\big)\\
&= [a] \triangleright \alpha \circ \mathit{singleton}_{WL} \circ u(\_ \vdash a) \circ \mathit{cosingleton}_L
\end{aligned}
$$

Output-related bottom-up phase:

$$
\begin{gathered}
s_4 = [\mathsf{I}] \qquad s_5 = [\mathsf{J}] \qquad s_6 = [\mathsf{K}] \qquad s_8 = [\mathsf{L}]\\
s_7 = [\mathsf{L}]^{\updownarrow}\\
s_2 = [\mathsf{I}] \mid [\mathsf{J}] \qquad s_3 = [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\\
s_1 = [\mathsf{I}] \mid [\mathsf{J}] \;/\; [\mathsf{K}] \mid [\mathsf{L}]^{\updownarrow}\\
s_0 = x
\end{gathered}
$$

Input-related top-down phase:

$$
\begin{aligned}
\mathit{cohwrap}_L\big(s_1, (n, e, s, w)\big) &= (n, s_1, s, s_1)\\
\mathit{coabove}_L\big(s_2, s_3, (n, s_1, s, s_1)\big) &= \big((n, s_2, s_3, s_2), (s_2, s_3, s, s_3)\big)\\
\mathit{cobeside}_L\big(s_4, s_5, (n, s_2, s_3, s_2)\big) &= \big((n_{\mathrm{a}}, s_5, s_6, s_5), (n_{\mathrm{b}}, s_4, s_7, s_4)\big)\\
&\qquad \text{where } (n_{\mathrm{a}}, n_{\mathrm{b}}) = \mathit{hsplit}(n, 1)\\
\mathit{cobeside}_L\big(s_6, s_7, (s_2, s_3, s, s_3)\big) &= \big((s_4, s_7, s_{\mathrm{a}}, s_7), (s_5, s_6, s_{\mathrm{b}}, s_6)\big)\\
&\qquad \text{where } (s_{\mathrm{a}}, s_{\mathrm{b}}) = \mathit{hsplit}(s, 1)\\
\mathit{covwrap}_L\big(s_7, (s_5, s_6, s_{\mathrm{b}}, s_6)\big) &= (s_7, s_6, s_7, s_6)
\end{aligned}
$$

Singletons again:

$$
\begin{aligned}
\mathit{cosingleton}(n_{\mathrm{a}}, s_5, s_6, s_5) &= (\mathsf{c}, \mathsf{J}, \mathsf{K}, \mathsf{J})\\
\mathit{cosingleton}(n_{\mathrm{b}}, s_4, s_7, s_4) &= (\mathsf{d}, \mathsf{I}, \mathsf{L}, \mathsf{I})\\
\mathit{cosingleton}(s_4, s_7, s_{\mathrm{a}}, s_7) &= (\mathsf{I}, \mathsf{L}, \mathsf{q}, \mathsf{L})\\
\mathit{cosingleton}(s_7, s_6, s_7, s_6) &= (\mathsf{L}, \mathsf{K}, \mathsf{L}, \mathsf{K})
\end{aligned}
$$

Transition-related bottom-up phase:

$$t_4(n_\mathrm{a}, s_5, s_6, s_5) = \left[u\left(\begin{smallmatrix} & \mathsf{c} & \\ \mathsf{J} & \mathsf{I} & \mathsf{J} \\ & \mathsf{K} & \end{smallmatrix}\right)\right] \qquad t_5(n_\mathrm{b}, s_4, s_7, s_4) = \left[u\left(\begin{smallmatrix} & \mathsf{d} & \\ \mathsf{I} & \mathsf{J} & \mathsf{I} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]$$

$$t_6(s_4, s_7, s_\mathrm{a}, s_7) = \left[u\left(\begin{smallmatrix} & \mathsf{I} & \\ \mathsf{L} & \mathsf{K} & \mathsf{L} \\ & \mathsf{q} & \end{smallmatrix}\right)\right] \qquad t_8(s_7, s_6, s_7, s_6) = \left[u\left(\begin{smallmatrix} & \mathsf{L} & \\ \mathsf{K} & \mathsf{L} & \mathsf{K} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]$$

$$t_7(s_5, s_6, s_\mathrm{b}, s_6) = \left[u\left(\begin{smallmatrix} & \mathsf{L} & \\ \mathsf{K} & \mathsf{L} & \mathsf{K} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]^{\updownarrow}$$

$$t_2(n, s_2, s_3, s_2) = \left[u\left(\begin{smallmatrix} & \mathsf{c} & \\ \mathsf{J} & \mathsf{I} & \mathsf{J} \\ & \mathsf{K} & \end{smallmatrix}\right)\right] \mid \left[u\left(\begin{smallmatrix} & \mathsf{d} & \\ \mathsf{I} & \mathsf{J} & \mathsf{I} \\ & \mathsf{L} & \end{smallmatrix}\right)\right] \qquad t_3(s_2, s_3, s, s_3) = \left[u\left(\begin{smallmatrix} & \mathsf{I} & \\ \mathsf{L} & \mathsf{K} & \mathsf{L} \\ & \mathsf{q} & \end{smallmatrix}\right)\right] \mid \left[u\left(\begin{smallmatrix} & \mathsf{L} & \\ \mathsf{K} & \mathsf{L} & \mathsf{K} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]^{\updownarrow}$$

$$t_1(n, s_1, s, s_1) = \left[u\left(\begin{smallmatrix} & \mathsf{c} & \\ \mathsf{J} & \mathsf{I} & \mathsf{J} \\ & \mathsf{K} & \end{smallmatrix}\right)\right] \mid \left[u\left(\begin{smallmatrix} & \mathsf{d} & \\ \mathsf{I} & \mathsf{J} & \mathsf{I} \\ & \mathsf{L} & \end{smallmatrix}\right)\right] \,/\, \left[u\left(\begin{smallmatrix} & \mathsf{I} & \\ \mathsf{L} & \mathsf{K} & \mathsf{L} \\ & \mathsf{q} & \end{smallmatrix}\right)\right] \mid \left[u\left(\begin{smallmatrix} & \mathsf{L} & \\ \mathsf{K} & \mathsf{L} & \mathsf{K} \\ & \mathsf{L} & \end{smallmatrix}\right)\right]^{\updownarrow}$$

$$t_0 = W^\gamma u(c \vdash x)$$

Hence we conclude $(W^\gamma u)^{\triangleright}(x) = (B^{\lambda^u}\alpha)?(x)$ and may invoke Theorem 4. $\square$